Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe

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Abstract

The Siklos class of solutions of Einstein's field equations is investigated by analytical methods. By studying the behaviour of free particles we reach the conclusion that the space-times represent exact gravitational waves propagating in the anti-de Sitter universe. The presence of a negative cosmological constant implies that the 'background' space is not asymptotically flat and requires a 'rotating' reference frames in order to fully simplify and view the behaviour of nearby test particles. The Kaigorodov space-time, which is the simplest representative of the Siklos class, is analyzed in more detail. It is argued that it may serve as a 'cosmological' analogue of the well-known homogeneous *pp*-waves in the flat universe.

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1 Introduction

The first class of exact solutions representing gravitational waves in general relativity was found by Brinkmann in 1923 [1]. The metrics were later discovered independently by several authors (cf. [2]) including Robinson who in 1956 recognized their physical meaning — these metrics are now known as *pp* -waves. In 1925, Beck [3] discovered cylindrical gravitational waves which were later studied by Einstein and Rosen [4]. In the beginning of the 1960's, the introduction of new geometrical concepts and methods (algebraic classification, gravitational ray optics, concept of the news function, spin coefficients etc.) had an important influence on finding new exact radiative solutions. These solutions, such as the plane-fronted waves [5]-[7] or the Robinson-Trautman 'spherical' waves [8], are now considered as standard 'prototypes' of exact gravitational waves.

An important step in treatment of gravitational radiation within the full non-linear general relativity was made by Penrose. His concept of a smoothly asymptotically flat space-time (cf. [9] and references therein) represents a rigorous geometrical framework for the discussion of gravitational radiation from *spatially isolated* sources. Moreover, the case of finite sources has an astrophysical relevance so that most of the work on gravitational radiation has been concerned with space-times which are either asymptotically flat (in some directions at least), cf. [10], or contain flat regions explicitly as, e.g., in the case of colliding plane gravitational waves (see [11] for a comprehensive review).

On the other hand, in the last two decades new exact solutions representing 'cosmological' gravitational waves in non-asymptotically flat models were found and analyzed, for example in [12]-[27] and elsewhere (cf. [15, 28, 29] for a review of the main works). Some of these solutions (usually admitting two spacelike Killing vectors) can be interpreted as spatially inhomogeneous models in which the homogeneity of the universe is broken due to gravitational waves. They may serve as exact models of the propagation of primordial gravitational waves and may be relevant for the (hypothetical) cosmological wave background.

In this work we concentrate on the physical interpretation of the class of exact type N solutions with a negative cosmological constant Λ found by Siklos [30]. In general, these solutions admit only one Killing vector. Moreover, since $\Lambda < 0$, we deal with gravitational waves 'in' an everywhere curved anti-de Sitter universe (the de Sitter and anti-de Sitter universes are the simplest natural cosmological 'background' space-times for solutions with Λ because they are conformally flat and they admit the same number of isometries as flat Minkowski space-time). Our real universe is probably not asymptotically flat and the whole theory of gravitational radiation should eventually be formulated with other boundary conditions than those corresponding to asymptotic flatness. Any exact explicit example of a wave propagating in a space-time which is not asymptotically flat may give a useful insight.

It may also serve as a 'test-bed' for numerical simulations.

In the next section we shall review the Siklos class of solutions. It will be shown to be identical with one subclass of space-times studied by Ozsváth, Robinson and Rózga [31]. In Section 3 we shall analyze the equation of geodesic deviation in frames parallelly transported along time-like geodesics. It will be demonstrated that this choice (although being the most natural one) is not suitable for a physical interpretation. The simple interpretation of the vacuum Siklos space-times (here by vacuum space-times we understand Einstein spaces with $\Lambda < 0$), presented in Section 4, will be given in rotating frames in which the solutions clearly represent exact gravitational waves propagating 'in' the anti-de Sitter universe. A surprising result is that the direction, in which the waves propagate, rotates with angular velocity $\omega = \sqrt{-\Lambda/3}$. In Section 5 the Kaigorodov solution [32], an interesting representative of the Siklos class, will be described (it is a homogeneous type N vacuum solution with $\Lambda < 0$ admitting 5 Killing vectors). The explicit form of all geodesics and a general solution of the equation of geodesic deviation in the Kaigorodov space-time will be presented in Section 6. Finally, remarks on its global structure will be given in Section 7.

2 The Siklos space-times

In 1985 Siklos found an interesting class of type N space-times for which the quadruple Debever-Penrose null vector field $\mathbf{k}$ is also a Killing vector field [30]. The metric can be written in the form

$$ds^2 = \frac{\beta^2}{x^2}(dx^2 + dy^2 + 2dudv + Hdu^2) \ , \tag{1}$$

where $\beta = \sqrt{-3/\Lambda}$, Λ is a negative cosmological constant, x and y are spatial coordinates, v is the affine parameter along the rays generated by $\mathbf{k} = \partial_v$, and u is the retarded time. If the vacuum equations (with $\Lambda < 0$) are to be satisfied, the function $H(x, y, u)$ must obey

$$H_{,xx} - \frac{2}{x}H_{,x} + H_{,yy} = 0 \ . \tag{2}$$

An explicit solution to this equation reads [30]

$$H = x^2 \frac{\partial}{\partial x}\left(\frac{f + \bar{f}}{x}\right) \equiv \frac{1}{2}(f_{,\zeta} + \bar{f}_{,\bar{\zeta}})(\zeta + \bar{\zeta}) - (f + \bar{f}) \ , \tag{3}$$

where $\zeta \equiv x + iy$ and $f(\zeta, u)$ is an arbitrary function, analytic in ζ. From (1) it is clear that the Siklos space-times are conformal to pp -waves. In fact, it was demonstrated in [30] that they represent the only non-trivial Einstein spaces conformal to non-flat pp -waves.

The metric (1) represents the anti-de Sitter solution when $H = 0$. Any particular solution of (2) can have an arbitrary profile $h(u)$. Therefore, as for pp -waves, sandwich waves can be obtained by taking h nonzero for a finite period of retarded time. In particular, by taking h to be a delta function, impulsive waves can be constructed [33].

Note that the Siklos class is identical with the special subclass $(IV)_0$ of non-twisting, non-expanding and shear-free space-times of the Kundt type found by Ozsváth, Robinson and Rózga [31] in the form

$$ds^2 = 2\frac{1}{p^2}d\xi d\bar{\xi} - 2\frac{q^2}{p^2}dU dV - \frac{q}{p}\,\tilde{H}(\xi,\bar{\xi},U)\,dU^2\ , \tag{4}$$

where $p = 1 + \frac{\Lambda}{6}\xi\bar{\xi}$ and $q = (1+\sqrt{-\frac{\Lambda}{6}}\xi)(1+\sqrt{-\frac{\Lambda}{6}}\bar{\xi})$. The limit $\Lambda \to 0$ in (4) with $\tilde{H}$ independent of Λ gives immediately the metric of pp -waves. The explicit transformation converting (4) to (1) is

$$\xi = -\sqrt{-\frac{6}{\Lambda}}\frac{(x+1/2)+iy}{(x-1/2)+iy}\ , \quad U = \frac{1}{\Lambda}\sqrt{\frac{3}{2}}u\ , \quad V = 12\sqrt{\frac{2}{3}}v\ , \tag{5}$$

so that $\tilde{H} = -4\Lambda H/x$. Before concluding this brief introductory section, let us present the Christoffel symbols for the metric (1) in coordinates $x^\mu = (v,x,y,u)$,

$$\begin{array}{llll}
\Gamma^0_{01} = -\dfrac{1}{x}\ , & \Gamma^0_{13} = \dfrac{1}{2}\,H_{,x}\ , & \Gamma^0_{23} = \dfrac{1}{2}\,H_{,y}\ , & \Gamma^0_{33} = \dfrac{1}{2}\,H_{,u}\ , \\
\Gamma^1_{03} = \ \dfrac{1}{x}\ , & \Gamma^1_{11} = -\dfrac{1}{x}\ , & \Gamma^1_{22} = \ \dfrac{1}{x}\ , & \Gamma^1_{33} = \dfrac{H}{x} - \dfrac{1}{2}\,H_{,x}\ , \\
\Gamma^2_{12} = -\dfrac{1}{x}\ , & \Gamma^2_{33} = -\dfrac{1}{2}\,H_{,y}\ , & \Gamma^3_{13} = -\dfrac{1}{x}\ , &
\end{array} \tag{6}$$

and all independent nonvanishing components of the Riemann tensor and the Weyl tensor,

$$\begin{array}{l}
R_{1013} = R_{2023} = R_{3003} = R_{1212} = F\ , \\
R_{1313} = \dfrac{1}{2}F(2H - xH_{,x} + x^2H_{,xx})\ , \quad R_{2323} = \dfrac{1}{2}F(2H - xH_{,x} + x^2H_{,yy})\ , \\
R_{1323} = \dfrac{1}{2}Fx^2H_{,xy} = C_{1323}\ , \quad C_{1313} = -C_{2323} = \dfrac{1}{4}Fx^2(H_{,xx} - H_{,yy})\ ,
\end{array} \tag{7}$$

where $F = -\beta^2/x^4$.

3 Particles in the Siklos space-times

In this section we derive an invariant form of the equation of geodesic deviation for the Siklos space-times which will be used in the next section for a physical interpretation. We consider an arbitrary test particle freely falling along timelike geodesic $x^\alpha(\tau)$ with τ being a proper time normalizing the particle four-velocity $u^\alpha = dx^\alpha/d\tau$ so that

$$u_\alpha u^\alpha = \epsilon\ , \tag{8}$$

where $\epsilon = -1$ (for $\epsilon = 0$ and $\epsilon = +1$ the geodesic would be null or spacelike, respectively). For the metric (1) the geodesic equations and equation (8) are

$$\ddot{v} \ = \ 2\dot{v}\frac{\dot{x}}{x} - Cx^2(\dot{x}H_{,x} + \dot{y}H_{,y}) - \frac{1}{2}C^2x^4H_{,u}\ ,$$

$$
\begin{aligned}
\left(\frac{\dot{x}}{x}\right)^{\cdot} &= -[2C\dot{v} + \frac{\dot{y}^2}{x^2} + C^2x^2(H - \frac{1}{2}xH_{,x})] \ , \\
\ddot{y} &= 2\dot{y}\frac{\dot{x}}{x} + \frac{1}{2}C^2x^4H_{,y} \ , \qquad \dot{u} \ = \ Cx^2 \ , \\
\left(\frac{\dot{x}}{x}\right)^2 &= -(2C\dot{v} + \frac{\dot{y}^2}{x^2} + C^2x^2H - \frac{\epsilon}{\beta^2}) \ ,
\end{aligned} \tag{9}
$$

where $\cdot \equiv d/d\tau$ and $C = const$ ($C \neq 0$ for $\epsilon = -1$ since otherwise $\dot{u} = 0$ would be in contradiction with (8)). Some particular solutions for a special form of H will be presented later on. However, for our purposes, it is not necessary to solve (9) explicitly. Our analysis will primarily be based on the equation of geodesic deviation

$$
\frac{D^2Z^\mu}{d\tau^2} = -R^\mu_{\alpha\beta\gamma}u^\alpha Z^\beta u^\gamma \ , \tag{10}
$$

an equation for a displacement vector $Z^\mu(\tau)$ connecting two neighbouring free test particles. In order to obtain an *invariant* relative motion we set up an orthonormal frame $\{\mathbf{e}_a\} = \{\mathbf{u}, \mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}\}$, $\mathbf{e}_a \cdot \mathbf{e}_b \equiv g_{\alpha\beta}e^\alpha_a e^\beta_b = \eta_{ab}$. By projecting (10) onto the frame we get

$$
\ddot{Z}^{(i)} = -R^{(i)}_{\ (0)(j)(0)}Z^{(j)} \ , \tag{11}
$$

where $Z^{(i)} \equiv e^{(i)}_\mu Z^\mu$ are frame components of the displacement vector and $\ddot{Z}^{(i)} \equiv e^{(i)}_\mu \frac{D^2Z^\mu}{d\tau^2}$ are relative accelerations. We start with a natural choice, namely a frame $\{\mathbf{e}_a(\tau)\}$ given by

$$
\begin{aligned}
e^\alpha_{(0)} &= u^\alpha = (\dot{v}, \dot{x}, \dot{y}, Cx^2) \ , \\
e^\alpha_{(1)} &= \sin\left(\frac{\tau}{\beta}\right)(\dot{v} + \frac{1}{\beta^2C}\ , \dot{x}, \dot{y}, Cx^2) - \cos\left(\frac{\tau}{\beta}\right)(\frac{1}{\beta C}\frac{\dot{x}}{x}\ , -\frac{x}{\beta}, 0, 0) \ , \\
e^\alpha_{(2)} &= \frac{x}{\beta}(-\frac{\dot{y}}{Cx^2}\ , 0, 1, 0) \ , \\
e^\alpha_{(3)} &= \cos\left(\frac{\tau}{\beta}\right)(\dot{v} + \frac{1}{\beta^2C}\ , \dot{x}, \dot{y}, Cx^2) + \sin\left(\frac{\tau}{\beta}\right)(\frac{1}{\beta C}\frac{\dot{x}}{x}\ , -\frac{x}{\beta}, 0, 0) \ ,
\end{aligned} \tag{12}
$$

that is *parallelly transported* along any timelike geodesic in the Siklos spacetime. Next step is to calculate the frame components of the Riemann tensor by using (7) and (12). Straightforward but somewhat tedious calculations give

$$
\begin{aligned}
R_{(1)(0)(1)(0)} &= -\frac{\Lambda}{3} + \mathcal{A}_+\cos^2\left(\frac{\tau}{\beta}\right) \ , & R_{(2)(3)(2)(3)} &= \frac{\Lambda}{3} - \mathcal{M}\cos^2\left(\frac{\tau}{\beta}\right) \ , \\
R_{(3)(0)(3)(0)} &= -\frac{\Lambda}{3} + \mathcal{A}_+\sin^2\left(\frac{\tau}{\beta}\right) \ , & R_{(1)(2)(1)(2)} &= \frac{\Lambda}{3} + \mathcal{M}\sin^2\left(\frac{\tau}{\beta}\right) \ , \\
R_{(1)(3)(1)(3)} &= \frac{\Lambda}{3} + \mathcal{A}_+ \ , & R_{(2)(0)(2)(0)} &= -\frac{\Lambda}{3} - \mathcal{M} \ , \\
R_{(1)(0)(2)(0)} &= R_{(1)(3)(2)(3)} = -\mathcal{A}_\times\cos\left(\frac{\tau}{\beta}\right) \ , && \\
R_{(2)(0)(3)(0)} &= R_{(1)(2)(1)(3)} = \mathcal{A}_\times\sin\left(\frac{\tau}{\beta}\right) \ , && \\
R_{(0)(1)(1)(3)} &= -\mathcal{A}_+\cos\left(\frac{\tau}{\beta}\right) \ , & R_{(0)(2)(2)(3)} &= \mathcal{M}\cos\left(\frac{\tau}{\beta}\right) \ ,
\end{aligned} \tag{13}
$$

$$R_{(0)(3)(1)(3)} = \mathcal{A}_+ \sin\left(\frac{\tau}{\beta}\right) , \qquad R_{(0)(2)(1)(2)} = -\mathcal{M} \sin\left(\frac{\tau}{\beta}\right) ,$$
$$R_{(1)(0)(3)(0)} = -\mathcal{A}_+ \sin\left(\frac{\tau}{\beta}\right) \cos\left(\frac{\tau}{\beta}\right) , \qquad R_{(1)(2)(2)(3)} = \mathcal{M} \sin\left(\frac{\tau}{\beta}\right) \cos\left(\frac{\tau}{\beta}\right) ,$$
$$R_{(0)(1)(1)(2)} = R_{(0)(3)(2)(3)} = -\mathcal{A}_\times \sin\left(\frac{\tau}{\beta}\right) \cos\left(\frac{\tau}{\beta}\right) ,$$
$$R_{(0)(1)(2)(3)} = \mathcal{A}_\times \cos^2\left(\frac{\tau}{\beta}\right) , \quad R_{(0)(3)(1)(2)} = \mathcal{A}_\times \sin^2\left(\frac{\tau}{\beta}\right) , \quad R_{(0)(2)(1)(3)} = \mathcal{A}_\times ,$$

where

$$\mathcal{A}_+ = -\frac{1}{2} C^2 x^5 \left(\frac{H_{,x}}{x}\right)_{,x} , \quad \mathcal{A}_\times = \frac{1}{2} C^2 x^5 \left(\frac{H_{,x}}{x}\right)_{,y} , \quad \mathcal{M} = \frac{1}{2} C^2 x^3 (x H_{,yy} - H_{,x}) . \quad (14)$$

By substituting the components (13) into (11) we get

$$\begin{aligned}
\ddot{Z}^{(1)} &= \frac{\Lambda}{3} Z^{(1)} - \mathcal{A}_+ \cos\left(\frac{\tau}{\beta}\right) \left[\cos\left(\frac{\tau}{\beta}\right) Z^{(1)} - \sin\left(\frac{\tau}{\beta}\right) Z^{(3)}\right] + \mathcal{A}_\times \cos\left(\frac{\tau}{\beta}\right) Z^{(2)} , \\
\ddot{Z}^{(2)} &= \frac{\Lambda}{3} Z^{(2)} + \mathcal{M} Z^{(2)} + \mathcal{A}_\times \left[\cos\left(\frac{\tau}{\beta}\right) Z^{(1)} - \sin\left(\frac{\tau}{\beta}\right) Z^{(3)}\right] , \qquad (15) \\
\ddot{Z}^{(3)} &= \frac{\Lambda}{3} Z^{(3)} + \mathcal{A}_+ \sin\left(\frac{\tau}{\beta}\right) \left[\cos\left(\frac{\tau}{\beta}\right) Z^{(1)} - \sin\left(\frac{\tau}{\beta}\right) Z^{(3)}\right] - \mathcal{A}_\times \sin\left(\frac{\tau}{\beta}\right) Z^{(2)} .
\end{aligned}$$

The structure of the equations is not simple. It may seem somewhat surprising because the Siklos solution is of Petrov type N so that it should describe gravitational waves affecting motions only in directions perpendicular to the direction of propagation, cf. [34]. However, equations (15) can be simplified with transverse effects becoming evident by a transformation from (12) to another frame. The idea follows naturally from the components of the quadruple Debever-Penrose vector $\mathbf{k} = \partial_v$, $k^{(1)} = \beta^2 C \sin\left(\frac{\tau}{\beta}\right)$, $k^{(2)} = 0$, $k^{(3)} = \beta^2 C \cos\left(\frac{\tau}{\beta}\right)$, which indicate that the *spacelike direction of propagation of the wave rotates uniformly in the* $(\mathbf{e}_{(1)}, \mathbf{e}_{(3)})$ *plane*. Thus, we can define a new frame $\{\mathbf{e}_{a'}\} = \{\mathbf{u}, \mathbf{e}_{(1')}, \mathbf{e}_{(2)}, \mathbf{e}_{(3')}\}$ by

$$\begin{aligned}
e^\alpha_{(1')} &= \cos\left(\frac{\tau}{\beta}\right) e^\alpha_{(1)} - \sin\left(\frac{\tau}{\beta}\right) e^\alpha_{(3)} = \left(-\frac{1}{\beta C}\frac{\dot{x}}{x}, \frac{x}{\beta}, 0, 0\right) , \\
e^\alpha_{(3')} &= \sin\left(\frac{\tau}{\beta}\right) e^\alpha_{(1)} + \cos\left(\frac{\tau}{\beta}\right) e^\alpha_{(3)} = \left(\dot{v} + \frac{1}{\beta^2 C}, \dot{x}, \dot{y}, C x^2\right) , \qquad (16)
\end{aligned}$$

in which the vector $\mathbf{k}$ has components $k^{(1')} = 0 = k^{(2)}$, $k^{(3')} = \beta^2 C \neq 0$. The orthonormal frame $\{\mathbf{e}_{a'}\}$ *is not parallelly transported* along any timelike geodesic since it rotates uniformly with respect to (12). Using (16) we can rewrite (15) as

$$\begin{aligned}
\ddot{Z}^{(1')} &= \frac{\Lambda}{3} Z^{(1')} - \mathcal{A}_+ Z^{(1')} + \mathcal{A}_\times Z^{(2)} , \\
\ddot{Z}^{(2)} &= \frac{\Lambda}{3} Z^{(2)} + \mathcal{M}\, Z^{(2)} + \mathcal{A}_\times Z^{(1')} , \qquad (17) \\
\ddot{Z}^{(3')} &= \frac{\Lambda}{3} Z^{(3')} .
\end{aligned}$$

This can be used for the interpretation of *general* Siklos space-times. In the following however, we concentrate only on vacuum solutions (with $\Lambda < 0$) describing 'pure' gravitational waves in the absence of matter.

4 Vacuum Siklos space-times as exact gravitational waves in the anti-de Sitter universe

Using the field equation (2) and its solution (3) we get

$$\begin{aligned}
\mathcal{A}_+(\tau) &= -\frac{1}{2}C^2x^5\left(\frac{H_{,x}}{x}\right)_{,x} \equiv -\frac{C^2}{32}(\zeta+\bar\zeta)^5\,\mathcal{R}e\{f_{,\zeta\zeta\zeta}\} = \mathcal{M}\ ,\\
\mathcal{A}_\times(\tau) &= +\frac{1}{2}C^2x^5\left(\frac{H_{,x}}{x}\right)_{,y} \equiv -\frac{C^2}{32}(\zeta+\bar\zeta)^5\,\mathcal{I}m\{f_{,\zeta\zeta\zeta}\}\ .
\end{aligned} \tag{18}$$

The system (17) with (18) represents the main result of our analysis. It is particularly well suited for the physical interpretation of vacuum Siklos space-times:

1. All test particles move isotropically one with respect to the other $\left(\ddot Z^{(i)} = \frac{\Lambda}{3}Z^{(i)}\ ,\ i=1',2,3'\right)$ if $\mathcal{A}_+ = 0 = \mathcal{A}_\times$, i.e. if $H(x,y,u) = c_0(u)+c_1(u)y+c_2(u)\,(x^2+y^2)$ corresponding to $f_{,\zeta\zeta\zeta}=0$. No gravitational wave is present in this case. This agrees with the fact that for H of this form the Siklos solution is conformally flat — the Weyl tensor vanishes (see (7)). The only conformally flat vacuum solution with $\Lambda<0$ is the anti-de Sitter spacetime, maximally symmetric solution of constant negative curvature. This explains the resulting isotropic motions. Thus, the terms proportional to Λ in (17) represent the influence of the *anti-de Sitter background.*

2. If the amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$ do not vanish (which is for $f_{,\zeta\zeta\zeta}\neq 0$) the particles are influenced (for $\Lambda\to 0$) similarly as by standard gravitational waves on Minkowski background (such as exact *pp* -waves or linearized waves [35]). However, for $\Lambda<0$ the influence of the gravitational wave adds with the anti-de Sitter isotropic background motions due to the presence of the Λ-terms. This supports our interpretation of the Siklos solution as an *exact gravitational wave in the anti-de Sitter universe.*

3. The gravitational *wave propagates in the spacelike direction of* $\mathbf{e}_{(3')}$ and has a *transverse character* since only motions in the perpendicular directions $\mathbf{e}_{(1')}$ and $\mathbf{e}_{(2)}$ are affected. The direction of propagation is *not* parallelly transported — it uniformly rotates with respect to parallel frames along any geodesic with angular velocity given by $\omega=\sqrt{-\Lambda/3}$. In the limit $\Lambda\to 0$ the effect of rotation vanishes.

4. The wave has *two polarization modes*: '+' and '×' with $\mathcal{A}_+$ and $\mathcal{A}_\times$ being the corresponding two independent *amplitudes.* The amplitudes given by (18) depend on the proper time of each particle through $x(\tau)$ and $H(x(\tau),y(\tau),u(\tau))$ where $x^\mu(\tau)$ describes the geodesic. Performing the rotation in the transverse plane

$$\tilde{\mathbf{e}}_{(1')} = \cos\vartheta\,\mathbf{e}_{(1')}+\sin\vartheta\,\mathbf{e}_{(2)}\ ,\quad \tilde{\mathbf{e}}_{(2)} = -\sin\vartheta\,\mathbf{e}_{(1')}+\cos\vartheta\,\mathbf{e}_{(2)}\ , \tag{19}$$

the motions are again given as in (17), only the amplitudes change according to

$$\tilde{\mathcal{A}}_+(\tau) = \cos 2\vartheta\, \mathcal{A}_+ - \sin 2\vartheta\, \mathcal{A}_\times \ , \quad \tilde{\mathcal{A}}_\times(\tau) = \sin 2\vartheta\, \mathcal{A}_+ + \cos 2\vartheta\, \mathcal{A}_\times \ . \tag{20}$$

Relations (20) represent the transformation (polarization) properties of the wave amplitudes. They are π-periodic so that the helicity is equal to 2. Moreover, by special choices of the polarization parameter $\vartheta = \vartheta_+$ or $\vartheta = \vartheta_\times$ one can set up at any event privileged frames in which either $\tilde{\mathcal{A}}_\times = 0$ or $\tilde{\mathcal{A}}_+ = 0$, i.e., the wave is purely polarized. Since $\vartheta_\times = \vartheta_+ + \frac{\pi}{4}$, the two modes are $\frac{\pi}{4}$ - shifted.

5. From (18) it follows that radiative vacuum Siklos space-times contain singularities at $\zeta+\bar{\zeta} = \infty$ (corresponding to $x = \infty$) since the components (13) of the Riemann tensor are proportional to diverging gravitational-wave amplitudes. According to definitions presented in [36, 37] there is a curvature singularity at $x = \infty$. Other singularities arise if $f_{,\zeta\zeta\zeta}$ in the amplitudes diverges.

We conclude this section by rewriting the equation of geodesic deviation. The form (17) is well suited for interpretation due to its simple structure but it is not useful for looking for solutions: $\ddot{Z}^{(i)} = e^{(i)}_\mu (D^2 Z^\mu / d\tau^2)$ is not a total time derivative of $Z^{(i)}(\tau)$ for $i = 1', 3'$ since $\mathbf{e}_{(1')}, \mathbf{e}_{(3')}$ are not parallelly transported. In fact,

$$\frac{d^2 Z^{(i)}(\tau)}{d\tau^2} = \ddot{Z}^{(i)} + 2\frac{De^{(i)}_\mu}{d\tau}\frac{DZ^\mu}{d\tau} + Z^\mu \frac{D^2 e^{(i)}_\mu}{d\tau^2} \ . \tag{21}$$

For (16) we get by using of $D\mathbf{e}_a/d\tau = 0$

$$\frac{D\mathbf{e}^{(1')}}{d\tau} = -\frac{1}{\beta}\mathbf{e}^{(3')} \ , \ \frac{D\mathbf{e}^{(3')}}{d\tau} = \frac{1}{\beta}\mathbf{e}^{(1')} \ , \ \frac{D^2\mathbf{e}^{(1')}}{d\tau^2} = -\frac{1}{\beta^2}\mathbf{e}^{(1')} \ , \ \frac{D^2\mathbf{e}^{(3')}}{d\tau^2} = -\frac{1}{\beta^2}\mathbf{e}^{(3')} \ . \tag{22}$$

Equations (21) thus take the form

$$\ddot{Z}^{(1')} = \frac{d^2 Z^{(1')}}{d\tau^2} + \frac{2}{\beta}\frac{dZ^{(3')}}{d\tau} - \frac{1}{\beta^2}Z^{(1')} \ , \qquad \ddot{Z}^{(3')} = \frac{d^2 Z^{(3')}}{d\tau^2} - \frac{2}{\beta}\frac{dZ^{(1')}}{d\tau} - \frac{1}{\beta^2}Z^{(3')} \ . \tag{23}$$

By combining (23) with (17) we get the following form of the equation of geodesic deviation

$$\begin{aligned}
\frac{d^2 Z^{(1')}}{d\tau^2} + \left(\frac{4}{\beta^2} + \mathcal{A}_+(\tau)\right) Z^{(1')} &= \mathcal{A}_\times(\tau) Z^{(2)} - \frac{2}{\beta}C_1 \ , \\
\frac{d^2 Z^{(2)}}{d\tau^2} + \left(\frac{1}{\beta^2} - \mathcal{A}_+(\tau)\right) Z^{(2)} &= \mathcal{A}_\times(\tau) Z^{(1')} \ , \\
Z^{(3')} &= \frac{2}{\beta}\int Z^{(1')}\, d\tau + C_1\tau + C_2 \ ,
\end{aligned} \tag{24}$$

C_1, C_2 being constants. The system can be integrated provided we know the explicit form of the geodesic $x^\mu(\tau)$ and therefore of the amplitudes $\mathcal{A}_+(\tau)$, $\mathcal{A}_\times(\tau)$. (In Section 6 we shall present a general solution of (24) for the case when $H = x^3$.) Let us note here only that

there always exists a trivial solution (along *any* timelike geodesic in *any* vacuum Siklos space-time) given by $Z^{(1')} = 0 = Z^{(2)}$, $Z^{(3')} = D$, D being a constant, i.e. (cf. (16)),

$$Z^{(1)} = D \sin\left(\frac{\tau}{\beta}\right), \quad Z^{(2)} = 0, \quad Z^{(3)} = D \cos\left(\frac{\tau}{\beta}\right) . \tag{25}$$

It has a simple interpretation: the particles may always *corotate uniformly in circles* with constant angular velocity $\omega = \sqrt{-\Lambda/3}$ around the 'fiducial' reference particle (if measured with respect to parallelly transported frames). For $\Lambda \to 0$ the rotation vanishes.

5 The Kaigorodov space-time

As an interesting particular example of the Siklos-type metric (1) we now analyze a vacuum solution with $\Lambda < 0$ given by $H = x^3$ corresponding to $f = \frac{1}{4}\zeta^3$ (cf. (3)),

$$ds^2 = \frac{\beta^2}{x^2}(dx^2 + dy^2 + 2du dv + x^3 du^2) . \tag{26}$$

In fact, such a solution represents the simplest non-trivial vacuum space-time of the Siklos type (for f quadratic in ζ one gets just the conformally flat anti-de Sitter space-time, cf. (18)). Therefore, it can be understood as a $\Lambda < 0$ analogue of the "homogeneous" *pp* -wave in Minkowski background [7] which is also the simplest vacuum *pp* space-time. The solution (26) was first discovered by Kaigorodov [32] in the form

$$ds^2 = (dx^4)^2 + e^{2x^4/\beta}[2dx^1 dx^3 + (dx^2)^2] \pm e^{-x^4/\beta}(dx^3)^2 . \tag{27}$$

Transformation between the Kaigorodov and the Siklos coordinates is given by

$$x^1 = \beta v , \quad x^2 = \beta y , \quad x^3 = \beta u , \quad x^4 = -\beta \ln |x| . \tag{28}$$

The solution has also been discussed independently in [38]-[40] and it is a special case of the $(IV)_0$ class found by Ozsváth, Robinson and Rózga (see (4) and Eq. (6.17) in [31]),

$$ds^2 = 2\frac{1}{p^2} d\xi d\bar{\xi} - 2\frac{q^2}{p^2} dU dV + \Lambda \frac{p}{q} dU^2 , \tag{29}$$

the transformation to (26) being given by (5). Other forms of the Kaigorodov solution can be found in [2], Eq. (10.33),

$$ds^2 = -\frac{12}{\Lambda} dZ^2 + 10k e^{2Z} dX^2 + e^{-4Z} dy^2 - 10U e^Z dZ dX - 2e^Z dU dX , \tag{30}$$

resulting from the transformation

$$x = \beta e^{2Z} , \quad u = -\sqrt{\frac{10k}{\beta^3}} X , \quad v = \sqrt{\frac{\beta^3}{10k}} e^{5Z} U , \tag{31}$$

and Eq. (33.2) (there is a misprint in [2]: the coefficient $2(\Lambda x)^{-2}$ should be $-3/(\Lambda x^2)$),

$$ds^2 = \frac{\beta^2}{x^2}(dx^2 + dy^2) - 2dU(dV + 2V\frac{dx}{x} - xdU) \ , \tag{32}$$

resulting from

$$u = \frac{\sqrt{2}}{\beta}U \ , \quad v = -\frac{1}{\sqrt{2}\beta}Vx^2 \ . \tag{33}$$

The Kaigorodov space-time is the only homogeneous type N solution (the quadruple Debever-Penrose vector being $\mathbf{k} = \partial_v$) of the Einstein vacuum field equations with $\Lambda \neq 0$ (necessarily with $\Lambda < 0$). It admits a five-parameter group of motions. The Killing vectors in the Siklos coordinates (v, x, y, u) are (see [30])

$$\begin{aligned} \xi^\mu_{(1)} &= (1,0,0,0) \ , & \xi^\mu_{(2)} &= (0,0,1,0) \ , & \xi^\mu_{(3)} &= (0,0,0,1) \ , \\ \xi^\mu_{(4)} &= (-y,0,u,0) \ , & \xi^\mu_{(5)} &= (5v,2x,2y,-u) \ , \end{aligned} \tag{34}$$

the corresponding isometries being: 1. $v' = v + v_0$, 2. $y' = y + y_0$, 3. $u' = u + u_0$, 4. $v' = -\frac{A^2}{2}u - Ay + v$, $y' = Au + y$, 5. $v' = e^{5B}v$, $x' = e^{2B}x$, $y' = e^{2B}y$, $u' = e^{-B}u$. From (34) we see that the quadruple Debever-Penrose null vector is also a Killing vector. However, it is not covariantly constant.

6 Particles in the Kaigorodov space-time

For $H = x^3$ representing the Kaigorodov solution, the equations of motion (9) give

$$\begin{aligned} \dot{x}^2 &= C^2x^7 - (2AC + B^2)x^4 + \epsilon\frac{x^2}{\beta^2} \ , \\ \dot{v} &= Ax^2 - Cx^5 \ , \qquad \dot{y} = Bx^2 \ , \qquad \dot{u} = Cx^2 \ , \\ x\ddot{x} - \dot{x}^2 &= \frac{5}{2}C^2x^7 - (2AC + B^2)x^4 \ , \end{aligned} \tag{35}$$

A, B, C being real constants of integration. Now we must distinguish two cases.

Case 1. If $\dot{x} = 0$ then the equations (35) give

$$\begin{aligned} v(\tau) &= \left(Ax_0^2 - Cx_0^5\right)\tau + v_0 \ , & x(\tau) &= x_0 \ , \\ y(\tau) &= Bx_0^2\tau + y_0 \ , & u(\tau) &= Cx_0^2\tau + u_0 \ , \end{aligned} \tag{36}$$

where v_0, x_0, y_0, u_0 and A, B, C are real constants satisfying the conditions $\frac{3}{2}C^2x_0^5\beta^2 = \epsilon$ and $\frac{5}{2}C^2x_0^3 = 2AC + B^2$. The first condition implies that for $x_0 < 0$, $C \neq 0$ all the geodesics are timelike ($\epsilon = -1$) and for $x_0 > 0$, $C \neq 0$ spacelike ($\epsilon = +1$). For $C = 0$ the geodesics are null ($\epsilon = 0$) and they have a simple form $v = Ax_0^2\tau + v_0$, $x = x_0$, $y = y_0$, $u = u_0$ since the second condition gives $B = 0$ in this case.

Case 2. If $\dot{x} \neq 0$ then the last equation in (35) can simply be omitted (since the first equation is its integral). The four remaining equations give

$$\begin{aligned}
\tau - \tau_0 &= \int \left(1/x\sqrt{C^2x^5 - (2AC + B^2)x^2 + \epsilon/\beta^2}\right) dx \ , \\
v(\tau) &= A\int x^2(\tau)\, d\tau - C\int x^5(\tau)\, d\tau + v_0 \ , \\
y(\tau) &= B\int x^2(\tau)\, d\tau + y_0 \ , \qquad u(\tau) \ = \ C\int x^2(\tau)\, d\tau + u_0 \ ,
\end{aligned} \tag{37}$$

where $\tau_0, v_0, y_0, u_0, A, B, C$ are arbitrary constants. For special values of the parameters the integrations can be performed analytically:

(i) If $C = 0$ then the geodesics must be spacelike. Their form is given either by

$$\begin{aligned}
v(\tau) &= A\frac{\beta}{2}\exp\left[2(\tau - \tau_0)/\beta\right] + v_0 \ , \\
x(\tau) &= \pm\exp\left[(\tau - \tau_0)/\beta\right] \ , \\
y(\tau) &= y_0 \ , \quad u(\tau) \ = \ u_0 \ ,
\end{aligned} \tag{38}$$

(for $B = 0$), or by

$$\begin{aligned}
v(\tau) &= \frac{A}{\beta B^2}\tanh\left[(\tau - \tau_0)/\beta\right] + v_0 \ , \\
x(\tau) &= \pm\left(\beta|B|\cosh\left[(\tau - \tau_0)/\beta\right]\right)^{-1} \ , \\
y(\tau) &= \frac{1}{\beta B}\tanh\left[(\tau - \tau_0)/\beta\right] + y_0 \ , \quad u(\tau) = u_0 \ ,
\end{aligned} \tag{39}$$

(for $B \neq 0$).

(ii) If $C \neq 0$ it is convenient to further simplify (37) by using the symmetries of the solution. For a Killing vector ξ^μ, the expression $u_\mu\xi^\mu$ is the constant of motion for any geodesic observer having the four-velocity u^μ. The Killing vectors $\xi^\mu_{(1)}$, $\xi^\mu_{(2)}$ and $\xi^\mu_{(3)}$ (see (34)) give relations embodied already in (35), the vectors $\xi^\mu_{(4)}$ and $\xi^\mu_{(5)}$ imply $Bu - Cy = const$ and $\dot{x}/x + \frac{5}{2}Cv + By - \frac{1}{2}Au = const$, respectively. These two relations simplify (37) into

$$\begin{aligned}
\tau - \tau_0 &= \int \left(1/x\sqrt{C^2x^5 - (2AC + B^2)x^2 + \epsilon/\beta^2}\right) dx \ , \\
u(\tau) &= C\int x^2(\tau)\, d\tau + u_0 \ , \quad y(\tau) \ = \ \frac{B}{C}u(\tau) + y_0 \ , \\
v(\tau) &= \frac{2}{5C}\left[\left(\frac{A}{2} - \frac{B^2}{C}\right)u(\tau) - \frac{\dot{x}(\tau)}{x(\tau)}\right] + v_0
\end{aligned} \tag{40}$$

(we have reparametrized the constants y_0 and v_0). In the case when $2AC + B^2 = 0$, the remaining two integrations can be performed explicitly: *null* geodesics are

$$v(\tau) \ = \ \frac{4}{25C}(\tau - \tau_0)^{-1} + \frac{B^2}{C|C|}\left[-\frac{5}{2}|C|(\tau - \tau_0)\right]^{1/5} - \frac{B^2}{2C^2}u_0 + v_0 \ ,$$

$$
\begin{aligned}
x(\tau) &= \left[-\frac{5}{2}|C|(\tau-\tau_0)\right]^{-2/5} , \\
y(\tau) &= -\frac{2B}{|C|}\left[-\frac{5}{2}|C|(\tau-\tau_0)\right]^{1/5} + \frac{B}{C}u_0 + y_0 , \qquad (41)\\
u(\tau) &= -\frac{2C}{|C|}\left[-\frac{5}{2}|C|(\tau-\tau_0)\right]^{1/5} + u_0 .
\end{aligned}
$$

Timelike geodesics are

$$
\begin{aligned}
v(\tau) &= -\frac{2}{5\beta C}\tan t \ - \frac{B^2}{2C^2}u(\tau) + v_0 , \\
x(\tau) &= (\beta C\cos t)^{-2/5} , \quad y(\tau) \ = \ \frac{B}{C}u(\tau) + y_0 , \qquad (42)\\
u(\tau) &= -2(\beta C\cos t)^{1/5} C_{-1}^{(3/5)}(\sin t) + u_0 ,
\end{aligned}
$$

where $t = 5\tau/2\beta + \tau_0$ and the symbol $C_n^{(b)}(z)$ denotes the Gegenbauer polynomial (generalized for negative values of n) which is a particular case of the hypergeometric function, $C_n^{(b)}(z) \equiv N_n^{(b)} F(-n, n+2b; b+1/2; (1-z)/2)$. (For $n>0$ the standard normalization coefficient is $N_n^{(b)} = \Gamma(n+2b)/(n!\Gamma(2b))$; in order to avoid singularities we define $N_n^{(b)} = 1$ for $n<0$. It can be shown that for *non-integer* a,

$$
\int_0^t \cos^a t\, dt = \frac{1}{1+a}\left(C_{-1}^{(1+a/2)}(0) - \cos^{1+a} t\, C_{-1}^{(1+a/2)}(\sin t)\right) , \qquad (43)
$$

which we have used in (42) for $a=-4/5$). Note that for $A = 0 = B$, the geodesics (42) are highly *privileged* since they are perpendicular ($u_\mu \xi^\mu = 0$) to the Killing vectors $\boldsymbol{\xi}_{(2)}$, $\boldsymbol{\xi}_{(3)}$. Moreover, for $y_0 = 0 = v_0$ they are perpendicular to $\boldsymbol{\xi}_{(4)}$, $\boldsymbol{\xi}_{(5)}$, too.

Finally, *spacelike* geodesics (40) for $2AC + B^2 = 0$ take the form

$$
\begin{aligned}
v(\tau) &= \frac{2}{5\beta C}\coth t - \frac{B^2}{2C^2}u(\tau) + v_0 , \\
x(\tau) &= (\beta C\sinh t)^{-2/5} , \quad y(\tau) \ = \ \frac{B}{C}u(\tau) + y_0 , \qquad (44)\\
u(\tau) &= \frac{2}{5}(\beta C)^{1/5}\int \sinh^{-4/5} t\, dt + u_0 .
\end{aligned}
$$

All the geodesics (41), (42) and (44) are in the region $x>0$. There are no null and timelike geodesics of the form (40) with $2AC + B^2 = 0$ in the region $x<0$; there are only spacelike geodesics

$$
\begin{aligned}
v(\tau) &= \frac{2}{5\beta C}\tanh t - \frac{B^2}{2C^2}u(\tau) + v_0 , \\
x(\tau) &= -(-\beta C\cosh t)^{-2/5} , \quad y(\tau) \ = \ \frac{B}{C}u(\tau) + y_0 , \qquad (45)\\
u(\tau) &= -\frac{2}{5}(-\beta C)^{1/5}\int \cosh^{-4/5} t\, dt + u_0 .
\end{aligned}
$$

We end this section by investigating the *relative* motion of particles in the Kaigorodov space-time. The amplitudes (18) are $\mathcal{A}_+ = -\frac{3}{2}C^2x^5$, $\mathcal{A}_\times = 0$ where $x \equiv x(\tau)$ depends on the *timelike* geodesic, i.e., it is given by (36), (40), or (42) in particular. The frame is now privileged since the wave looks purely '+' polarized. Therefore, (24) decouples into the 'Schrödinger-type' equations, $\mathcal{A}_+$ being a 'potential', and explicit solutions can be found:

Case 1. If $\dot{x} = 0$ then the geodesics are given by (36) with $\epsilon = -1$ implying $\mathcal{A}_+ = -\frac{3}{2}C^2x_0^5 = 1/\beta^2$. In this case the general solution of (24) is

$$\begin{aligned}
Z^{(1')} &= D\cos\Big(\frac{\sqrt{5}}{\beta}\tau+\tau_0\Big)-\frac{2}{5}\beta C_1\ , \\
Z^{(2)} &= E_1\tau+E_2\ , \\
Z^{(3')} &= \frac{2}{\sqrt{5}}D\sin\Big(\frac{\sqrt{5}}{\beta}\tau+\tau_0\Big)+\frac{1}{5}C_1\tau+C_2\ ,
\end{aligned} \tag{46}$$

C_1, C_2, D, E_1, E_2 being real constants. The test particles move with a constant velocity E_1 in the direction of $\mathbf{e}_{(2)}$ whereas in the $(\mathbf{e}_{(1')}, \mathbf{e}_{(3')})$ plane the particles move in *ellipses*. In particular, for $C_1 = D = E_1 = E_2 = 0$ we get the solution (25) describing uniform rotations.

Case 2. If $\dot{x} \neq 0$ then the timelike geodesics are given by (40) with $\epsilon = -1$. Using (35) one can verify that the solution of (24) is

$$\begin{aligned}
Z^{(1')} &= \frac{\dot{x}}{x^3}\Big(D_1+\int\frac{x^4}{\dot{x}^2}(D_2x^2+\frac{C_1}{\beta})\,d\tau\Big)\ , \\
Z^{(2)} &= \frac{1}{x}\,\Big(E_1+E_2\int x^2\,d\tau\Big)\ , \\
Z^{(3')} &= \frac{2}{\beta}\int Z^{(1')}\,d\tau+C_1\tau+C_2\ ,
\end{aligned} \tag{47}$$

where $x \equiv x(\tau)$ follows from (40) and is unique for any geodesic. Since (47) contains 6 independent constants of integration it is a *general solution* of the equation of geodesic deviation. As in the previous case, for $C_1 = D_1 = D_2 = E_1 = E_2 = 0$ we get the solution (25). In particular, for the geodesics (42) with $A = 0 = B$ we have $x(\tau) = (\beta C\cos t)^{-2/5}$ for which (47) can be integrated into

$$\begin{aligned}
Z^{(1')} &= D_1\sin t\cos^{-1/5}t+D_2\cos^{6/5}t\,C^{(6/5)}_{-2}(\sin t)-\frac{2}{11}\beta C_1\cos^2 t\,C^{(8/5)}_{-2}(\sin t)\ , \\
Z^{(2)} &= E_1\cos^{2/5}t+E_2\cos^{3/5}t\,C^{(3/5)}_{-1}(\sin t)\ , \\
Z^{(3')} &= \frac{4}{5}\int Z^{(1')}\,dt+C_1\tau+C_2
\end{aligned} \tag{48}$$

(we have used (43), the identity $C^{(b)}_{-2}(z) \equiv (2b-1)-(2b-2)zC^{(b)}_{-1}(z)$, and we have re-defined the constants of integration). Notice that for $A = 0 = B$ the amplitude $\mathcal{A}_+ = (\Lambda/2)\cos^{-2}t$ is independent of the geodesic (given by some value of C) and that $\mathcal{A}_+ \to 0$ as $\Lambda \to 0$. In such a limit equations (24) reduce to $d^2Z^{(i)}/d\tau^2 = 0$, $i = 1', 2, 3'$. Considering also $(\mathbf{e}_{(1')}, \mathbf{e}_{(3')}) \to (\mathbf{e}_{(1)}, \mathbf{e}_{(3)})$ as $\Lambda \to 0$, cf. (16), we conclude that test particles move uniformly as in flat Minkowski space.

7 On the global structure of the Kaigorodov space-time

In this section we briefly touch upon some global properties of the Kaigorodov space-time. The metric (26) indicates that the space-time is regular everywhere except at $x = 0$ and $x = \infty$. All components of the curvature tensor for the Kaigorodov solution in the orthonormal frame parallelly propagated along time-like geodesics are given by (13) with $\mathcal{A}_+ = \mathcal{M} = -\frac{3}{2}C^2x^5$, $\mathcal{A}_\times = 0$. Therefore, $x = 0$ represents only a coordinate singularity. However, there is a *curvature singularity at* $x = \infty$; according to [37], or the classification scheme introduced in [36], it is a "p.p. curvature singularity", or a "C^0 curvature singularity", respectively. Geodesic observers moving along the time-like geodesics (40), (42) inevitably reach the singularity $x = +\infty$ in a finite proper time ($\Delta\tau \sim \int_{x_0}^{\infty} x^{-7/2}\, dx \sim [x^{-5/2}]_{x_0}^{\infty} = const < \infty$), whereas observers $x = x_0 < 0$ moving along (36) escape (there are no timelike observers $x = x_0 > 0$).

On the other hand, the region $x = 0$ can not be reached by any time-like observer (cf. (35) where $\dot{x}^2 \sim -x^2/\beta^2$ as $x \to 0$ would be a contradiction). This indicates that $x = 0$ represents the null and/or spacelike infinity. Indeed, the transformation

$$\begin{aligned} \eta &= \beta\cos(T/\beta)/\mathcal{D}\ , \qquad & x &= \beta\cos\chi/\mathcal{D}\ , \\ y &= \beta\sin\chi\cos\vartheta/\mathcal{D}\ , \qquad & z &= \beta\sin\chi\sin\vartheta\cos\varphi/\mathcal{D}\ , \end{aligned} \tag{49}$$

where $\eta = (u-v)/\sqrt{2}$, $z = (u+v)/\sqrt{2}$ and $\mathcal{D} = \sin(T/\beta) + \sin\chi\sin\vartheta\sin\varphi$, brings the metric (26) into the form

$$\begin{aligned} ds^2 &= \frac{\beta^2}{\cos^2\chi}\left\{-\frac{dT^2}{\beta^2} + d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right\} \\ &+ \frac{\beta^5}{2}\frac{\cos\chi}{\mathcal{D}^5}\Big\{-[1+\cos(T/\beta-\varphi)\sin\chi\sin\vartheta]\frac{dT}{\beta} + \sin(T/\beta-\varphi)\cos\chi\sin\vartheta\, d\chi \\ &\qquad + \sin(T/\beta-\varphi)\sin\chi\cos\vartheta\, d\vartheta - \sin\chi\sin\vartheta[\cos(T/\beta-\varphi)+\sin\chi\sin\vartheta]\, d\varphi\Big\}^2\ . \end{aligned} \tag{50}$$

This form of the Kaigorodov space-time shows explicitly that the metric approaches asymptotically the anti-de Sitter metric in standard global coordinates as $x \to 0$, i.e., $\chi \to \pm\pi/2$ (cf. §5.2 in [37] where $\cosh r = 1/\cos\chi$). In the literature, these coordinates are used for the construction of the Penrose diagram of the anti-de Sitter space-time — choosing a conformal factor $\Omega = \frac{1}{\beta}\cos\chi$, the boundary of the anti-de Sitter manifold given by $\Omega = 0$ ($\chi = \pm\pi/2$) represents null and spacelike infinity which can be thought of as a *timelike* surface since $\Omega_{,\alpha}\Omega^{,\alpha} > 0$ at $\Omega = 0$. Using the same conformal factor for the Kaigorodov space-time (50) we conclude that the boundary $\Omega = 0$ corresponding to $x = 0$ represents an 'anti-de Sitter-like' scri having the topology $R \times S^2$. Therefore, the Kaigorodov space-time is *weakly asymptotically anti-de Sitter* according to the definition given in [41]. This is not true, of course, in regions where $\mathcal{D} = 0$ representing singularities $x = \pm\infty$.

The fact that $x = 0$ is an infinity for null and spacelike observers is supported by the asymptotic behavior of the geodesics described by the equation (37). For spacelike geodesics ($\epsilon = +1$) we get $|x| \sim \exp(\Delta\tau/\beta)$ as $x \to 0$. Similarly, null geodesics ($\epsilon = 0$) are asymptotically $x \sim (\Delta\tau)^{-2/5}$ (cf. (41)), or $|x| \sim (\Delta\tau)^{-1}$ (for $2AC + B^2 < 0$). Therefore, the boundary $\Omega = 0$ is reached only at infinite value of the affine parameter τ of null and spacelike geodesics.

An important consequence of this is that the Kaigorodov space-time (26) splits, in fact, into two disjointed regions $x > 0$ and $x < 0$ (corresponding to different signs in (27)) which can not mutually communicate. These two regions are quite different. For example, the singularity $x = +\infty$ is reached by most of the timelike, null, and spacelike geodesics in $x > 0$, the singularity $x = -\infty$ is reached by *only one* (spacelike) geodesic $x(\tau) = -\exp[(\tau - \tau_0)/\beta]$ (cf. (38)) as $\tau \to \infty$. Also, the geodesics (36) given by $x = x_0$ are spacelike or null in $x > 0$ whereas they are timelike and null in $x < 0$. Moreover, $g_{\mu\nu}\xi^\mu_{(3)}\xi^\nu_{(3)} = \beta^2 x$ so that the Killing vector $\xi_{(3)} = \partial_u$ (see (34)) is spacelike for $x > 0$ but timelike for $x < 0$. This implies that the Kaigorodov space-time is *stationary* in $x < 0$ with u being a time coordinate.

Note that, surprisingly, the stationarity does not exclude the presence of gravitational radiation. For example, standard "homogeneous" (or "plane") *pp*-waves [2, 7] — a 'textbook model' of exact gravitational waves — are given by the metric $ds^2 = 2d\xi d\bar{\xi} - 2dudv - (g + \bar{g})du^2$, where $g = A(u)\xi^2$. Considering the simplest case, $A = 1$, and introducing real spacelike coordinates x and y by $\sqrt{2}\xi = x + iy$, we get $ds^2 = dx^2 + dy^2 - 2dudv - (x^2 - y^2)du^2$. This space-time is also stationary in the regions where $|x| > |y|$ since the Killing vector $\xi = \partial_u$ is timelike: it represents a radiation with a constant amplitude rather than a periodic-like gravitational wave.

8 Concluding remarks

Our study of the Siklos class of exact solutions indicates that a reasonable physical interpretation of these space-times can be given if one investigates the equation of geodesic deviation in the suitable frame. As in the linearized theory, exact waves of the Siklos type manifest themselves by typical effects on particle motions (transversality and specific polarization properties). The space-times describe exact gravitational waves propagating in the anti-de Sitter universe. Somewhat surprising is that, due to the presence of a negative cosmological constant, the direction of propagation of the waves rotates.

The analysis provided independent arguments for the interpretation suggested by Siklos [30] of the space-times (1) as 'Lobatchevski *pp*-waves' (Siklos proved that they admit a foliation of totally geodesic two-dimensional spacelike surfaces of constant negative curvature which are the wave surfaces of gravitational waves) and also for Ivor Robinson's suggestion

[40] that the Kaigorodov solution may be interpreted as a 'plane-fronted gravitational wave against the anti-de Sitter background' (since the group of isometries is five parametric). Therefore, the Siklos space-times — and the Kaigorodov solution in particular — can be understood as natural cosmological ($\Lambda < 0$) analogues of the *pp*-waves in the flat universe.

We hope that the studied space-times can be used in numerical relativity as the test beds for numerical codes aimed to understand realistic situations in more general cosmological contexts.

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